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Problem:

Lex $X := (X_1, ..., X_n)^t \sim \mathcal{N}_n(0, \Sigma)$ and $U := max\{X_1, .., X_n\}$. Find an upper bound on $\mathbb{E}(U)$.

Solution:

Observe that

$$e^{t\mathbb{E}U} \leq \mathbb{E}e^{tU} = \mathbb{E}\underset{i}{max}\{e^{tY_i}\} \leq \sum_1^n \mathbb{E}e^{tY_i} = \sum_1^n M_{Y_i}(t) \leq \sum_1^n e^{t^2\frac{\sigma^2}{2}} = ne^{t^2\frac{\sigma^2}{2}}$$

where $M_{Y_i}(t)$ denotes the moment generating function of Y_i evaluated at t and $\sigma^2 := \underset{i}{max}\{(\Sigma)_{ii}\}$. Note, the first inequality is a consequence of the convexity of e^x and Jensen's inequality. The subsequent equality follows from the fact that e^x is monotone increasing.

Taking logs on both sides of this inequality and dividing by t (assuming $t \neq 0$) gives

$$\mathbb{E}U \leq \frac{\log(n)}{t} + \frac{t\sigma^2}{2} := g(t).$$

Thus, to find the tightest bound, it suffices to find the global minimum of g

$$g'(t) = \frac{\log(n)}{t^2} + \frac{\sigma^2}{2} \overset{set}{=} 0 \implies t_{opt} = (\frac{2\log(n)}{\sigma^2})^{\frac{1}{2}}.$$

Plugging t_{opt} into g and some algebra reveals that

$$g(t_{opt}) = \sqrt{2\sigma^2 \log(n)}$$

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Therefore,

$$\mathbb{E}U \leq \sqrt{2\sigma^2 \log(n)}$$

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